ARTICLE NO. 78-1412R

Minimum and Maximum Endurance Trajectories for Gliding Flight in a Horizontal Plane

Walton E. Williamson Jr.
Sandia Laboratories, Albuquerque, New Mex.

The problem of determining minimum and maximum endurance trajectories for subsonic gliding flight in a horizontal plane is considered. These trajectories are of interest for the low-level delivery of weapon systems where it is desirable to have a weapon remain in the air, yet close to the ground, while the aircraft delivering the weapon leaves the area. If the weapon is placed inside a small lifting vehicle, then the problem is to determine either minimum or maximum endurance trajectories for the vehicle that return to a designated target. This problem is formulated and numerical results are computed for a small lifting vehicle.

Nomenclature

a	= vector of parameters to be determined by the optimization program
B	= constant = $0.5\,\rho\,S/m$
C_L	= lift coefficient
C_D	= drag coefficient
C_{D0}	= zero lift drag coefficient
C_{Lh}	= maximum value of lift coefficient in the horizontal plane
D	= drag force = $0.5\,\rho S V^2 C_D$
f	= vector of first derivatives of state equations
g	= gravitational acceleration
G	= performance index, scalar quantity to be minimized
K	= induced drag factor
L	= lift force = $0.5\,\rho\,S V^2 C_L$
m	= mass of vehicle
M	= vector of terminal constraints
N	= total number of parameters associated with optimization problem = $q+2$
p	= polynomial representation of control variable
q	= order of polynomial representation of control
S	= planform area of wing
t	= time, independent variable
u	= control variable
V	= magnitude of velocity
W	= weight of vehicle
x,y,x	= position coordinates of vehicle
ρ	= density of atmosphere
ψ	= velocity yaw angle
μ	= bank angle
λ	= exponent of C_L in induced drag term
λ_ψ,λ_V	= Lagrange multipliers associated with the state equations
τ	= normalized independent variable
$(\dot{\ })$	= first derivative with respect to time d/dt

Subscripts

0	= evaluated at initial time
f	= evaluated at final time
s	= specified numerical value

Presented as Paper 78-1412 at the AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 7-9, 1978; submitted Aug. 31, 1978; revision received March 30, 1979.

Index categories: Guidance and Control; Analytical and Numerical Methods.

*Member of Technical Staff, Aeroballistics Projects Div. Member AIAA.

Introduction

IT is often desirable to have an aircraft fly at very low altitudes when delivering weapons. It then becomes necessary to have some delay between the time a weapon is released and the explosion, for safe escape of the airplane. It is also desirable to initially fly over the target area and to keep the weapon at low altitudes after it has been released. One way of accomplishing this is to place the weapon inside a small lifting vehicle. The small vehicle is released and then flies some trajectory in a horizontal plane before returning to the target. It is desirable to consider only gliding flight so that thrusting devices will not be required by the small vehicle. The small vehicle should remain in the air as long as possible to allow the aircraft that dropped the weapon to get as far away from the target as possible. Thus, maximum endurance gliding trajectories that remain in a horizontal plane and return to a specified target are of interest. For some missions, it is desirable to get the vehicle to the target as rapidly as possible. Thus, minimum time flights are also of interest.

This paper is concerned with the numerical determination of trajectories of this type. Aerodynamic characteristics of a small lifting vehicle that could be carried by a larger airplane are used in the numerical studies. Both minimum and maximum endurance trajectories are numerically calculated for this specific vehicle.

Several other papers have investigated optimal turning trajectories. References 1-3 have investigated optimal turning trajectories for a thrusting vehicle. Reference 4 considered a gliding vehicle, but was interested in three-dimensional turning trajectories for a re-entry vehicle. This paper, as mentioned before, investigates optimal horizontal trajectories for a gliding vehicle. Both maximum and minimum time trajectories are presented.

Analysis

Gliding flight in a horizontal plane is described by a solution to the following equations:

$$\dot{x} = V\sin\psi \tag{1a}$$

$$\dot{y} = V\cos\psi \tag{1b}$$

$$\dot{V} = -D/m \tag{1c}$$

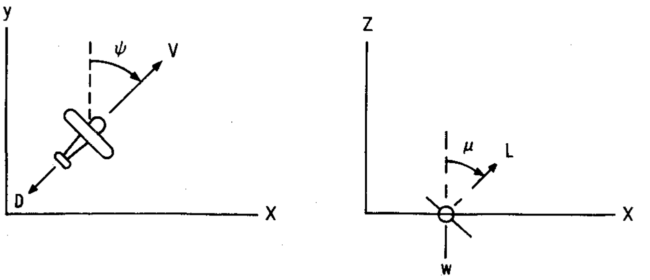

Fig. 1 Geometry for horizontal flight.

$$\dot{\psi} = L\sin\mu / mV \tag{1d}$$

$$L\cos\mu = W \tag{1e}$$

$$C_D = C_{D0} + K C_L^{\lambda} \tag{1f}$$

These are derived in Ref. 5 and the geometry is shown in Fig. 1.

There are two control variables for the problem. Both the bank angle and lift coefficient may be varied. They may not be varied independently, however, as Eq. (1e) shows. It is possible to use this equation to eliminate one of the control variables from the differential equations. Then either μ or C_L is treated as the control. The technique used to solve this problem will involve guessing a control history and iteratively correcting it until an optimal solution is found. During the iteration procedure, reasonably large changes in the control are desirable if the method is to converge in a reasonable number of iterations. The control variable should be selected so that large changes may be calculated for the control without causing numerical difficulties. If μ is used as the control, then $\dot{\psi}$ goes to infinity as μ approaches 90 deg. If C_L is used as the control, then $\dot{\psi}$ becomes imaginary for some values of C_L. Thus the choice of either C_L or μ as the control presents possible numerical difficulties. This is avoided if the lift in the horizontal plane is treated as the control. Thus choose

$$C_L\sin\mu = C_{Lh}\sin p \tag{2}$$

where p is the control variable. The variable C_{Lh} is the maximum value of the lift coefficient in the horizontal plane. Then.

$$C_L = [C_{Lh}^2\sin^2 p + g^2/(B^2V^4)]^{1/2} \tag{3}$$

which is used in Eq. (1f) to determine C_D. This choice for the control allows p to change substantially during the iterative procedure. It does not introduce possible singularities into the differential equations. It does not bound C_L directly, but a judicious choice for C_{Lh} will keep the maximum value of C_L near a prescribed boundary. Note that an inequality constraint for C_L would probably be necessary since minimum time trajectories are expected to require very large values for the lift coefficient. Equation (2) bounds the horizontal component of C_L. This should be the largest component for minimum time trajectories. Thus, this choice for the control also eliminates the necessity of imposing inequality constaints directly. The optimization problem then requires the determination of p to fly the vehicle from a specified initial state to a specified set of terminal conditions while extremizing some quantity.

The general optimization problem then may be stated as follows:

minimize

$$G(x_f, t_f) \tag{4}$$

subject to

$$x_0 = x_{0s} \qquad t_0 = t_{0s}$$
$$\dot{x} = f(x,u,t)$$
$$M(x_f, t_f) = 0 \tag{5}$$

For these equations, x is an n-dimensional state vector and should not be confused with the scaler variable used as one of the coordinates.

There are many numerical methods capable of calculating the optimal trajectories previously described. The one used here is described in Ref. 6. The control is parametrized. In this case, polynomials are used. Thus choose

$$p = a_0 + a_1\tau^2 + \ldots a_q\tau^q \tag{6}$$

where the a's are constants to be determined and τ is the normalized time, $t = a_{q+1}\tau(0 \le \tau \le 1)$. The optimization problem becomes a parameter optimization problem rather than a calculus of variations problem. It is necessary to determine the parameters $a_0, a_1, \ldots a_{q+1}$ that specify the optimal trajectory rather than the control μ, which is a function of time. It should be noted that requiring u to have a specific functional form reduces the class of admissable controls. Thus not all control histories can be approximated by a fairly low-order polynomial. Most optimal control problems shown in the literature, however do have fairly simple control shapes. Also, if the optimal control history is very complex, it will probably be impossible to implement. Hence, if q is chosen to be a small number such as 5 or 6, it is expected that a reasonably simple control history, which will be a fairly good representation of the true optimal control, will be obtained.

Two techniques are used to compute the a vector. The first is a gradient projection method similar to the ones described in Refs. 6 and 7. This approach is good at satisfying terminal conditions and reasonably good at finding the optimal value of the performance index. This approach is usually terminated before the true optimal values for the parameters are determined. Once the control is parameterized, however, the emphasis is on calculating good approximations to the true optimal. This is consistent with the use of the gradient scheme.

The second scheme used is a second-order or Newton-type method. This scheme causes the first derivatives of the augmented performance to be zero. Thus solutions computed using this scheme satisfy the first-order necessary conditions for a minimizing solution. This scheme is used to refine the answers obtained using the gradient scheme and to study the effect of changing the order of the polynomial approximation to the control.

Four different minimization problems are considered for the gliding aircraft. In all cases the initial conditions are fixed. Numerical values used are $x_0 = y_0 = \psi_0 = 0$ and $V_0 = V_{0s}$. The performance index and end conditions are changed in the following manner:

$$\text{Case 1:} \quad G = t_f \qquad M^T = [x_f, y_f] \tag{7}$$

$$\text{Case 2:} \quad G = t_f \qquad M^T = [x_f] \tag{8}$$

$$\text{Case 3:} \quad G = -t_f \qquad M^T = [x_f, y_f, V_f - V_{fs}] \tag{9}$$

$$\text{Case 4:} \quad G = -t_f \qquad M^T = [x_f, V_f - V_{fs}] \tag{10}$$

Cases 1 and 2 are minimum time problems. Case 1 requires the vehicle to return to the origin or the spot where separation occurs. Case 2 requires the vehicle to return to the line of flight. Cases 3 and 4 are maximum time problems. End conditions here are the same as cases 1 and 2, respectively, except that the final velocity is also specified. Since these are maximum endurance trajectories, a terminal velocity must be specified. The terminal velocity should either be the stall velocity or a value higher than the stall velocity. This keeps the computer program from calculating extremely large values for C_L and unrealistic flight conditions, near the end of the trajectory. A value is not specified for V_f in cases 1 and 2 since V_f is expected to be substantially above the stall velocity for the minimum time problem.

Results

The optimization technique described earlier was used to numerically compute the solutions to the four cases described in the previous section. The results are shown in Figs. 2-13 and described in this section.

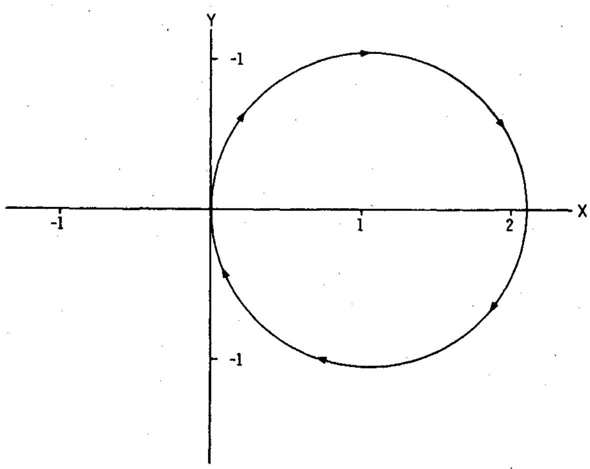

Fig. 2 Trajectory for case 1 ($N=6$, grad) (1 unit = 1000 ft).

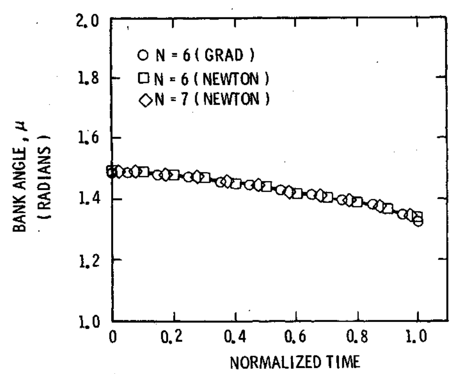

Fig. 3 Bank angle vs time for case 1.

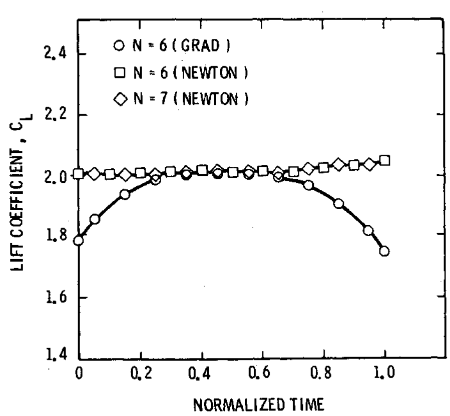

Fig. 4 Lift coefficient vs time for case 1.

All trajectories shown turn initially to the right. This is a result of initial guesses for the control history that cause the vehicle to turn in this direction. Optimal trajectories that turn to the left initially could have been calculated. Due to symmetry in the problem, however, solutions that turn to the left would be equivalent to those shown if the ones shown were rotated 180 deg about the y axis.

The initial and final velocities used for the vehicle are 700 ft/s and 250 ft/s, respectively. The numerical values used for the constants associated with the vehicle are shown in Table 1. These are reasonable values for a small lifting vehicle such as some of the current RPV's. For the minimum time problem,

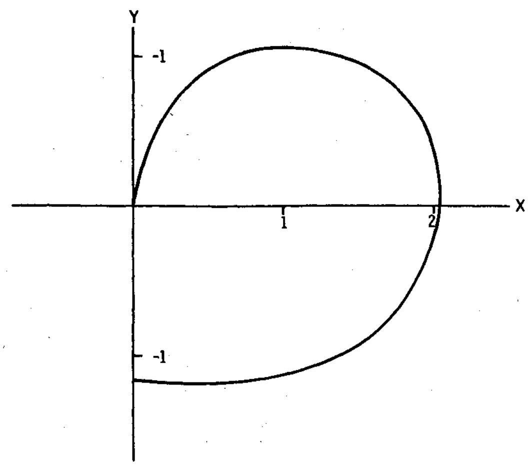

Fig. 5 Trajectory for case 2 ($N=6$, grad) (1 unit = 1000 ft).

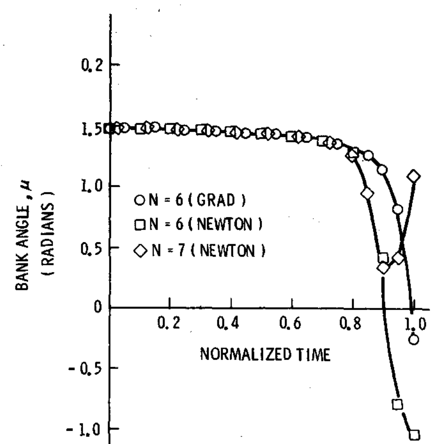

Fig. 6 Bank angle vs time for case 2.

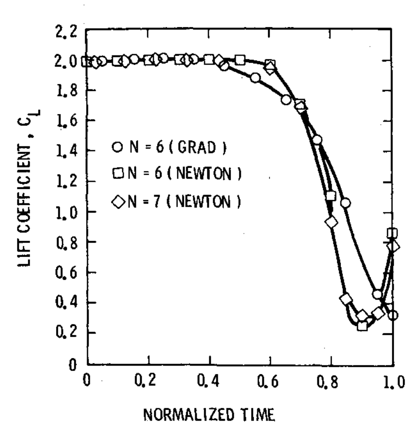

Fig. 7 Lift coefficient vs time for case 2.

Table 1 Values of constants used in optimization program

S/m	$=0.402175$ ft^2/slug
C_{D0}	$=0.015$
K	$=0.042$
ρ	$=0.0023769$ slug/ft^3
V_{0s}	$=700$ ft/s
V_{fs}	$=250$ ft/s
λ	$=2$

$C_{Lh}=2$, and for the maximum time problem, $C_{Lh}=1$. Solutions for all of the problems were initially obtained using the gradient method with $q=4$ or $N=6$. These results were used to start the Newtonian solution to the problem. Then the number of parameters was increased to seven, and the problem was resolved using the Newton method. This allows a comparison of the results obtained using the gradient scheme with a solution which satisfies the first-order necessary conditions. It also allows the effect of increasing the order of the polynomial control approximation to be seen.

The results for the minimum time problem are shown in Figs. 2-7. Case 1 is shown in Figs. 2-4. Only one trajectory is shown since the three trajectories are essentially the same. Tabulated listings of the a vector for the three solutions are shown in Tables 2-4. The general characteristic of the optimal solution is a nearly circular trajectory in the right-hand side of the plane. The radius of the circle is gradually decreasing as time increases. The lift coefficient remains near the value of C_{Lh}. This indicates that most of the lift is in the horizontal plane, as can be seen in Fig. 3. The bank angle is almost 90 deg initially and rolls up slightly as time increases. The decrease in bank angle is necessary to maintain a constant vertical force since the velocity also decreases along the trajectory. Thus for case 1, the vehicle flies near the maximum value of C_L. It banks the turn as much as possible while maintaining horizontal flight and flies a nearly circular return to the origin.

The comparison of the three solutions shows that there is very little change in G between them. Thus the gradient scheme provides a good estimate of the optimal value of G and increasing the order of the control approximation does not substantially alter the solution. The gradient scheme does not properly shape all of the C_L history (Fig. 4) since it does not cause it to be on the boundary for all of the trajectory.

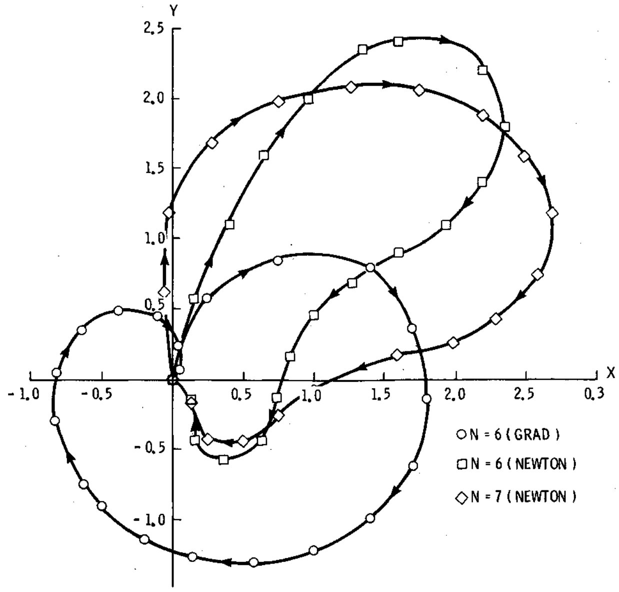

Fig. 8 Trajectories for case 3.

Table 2 Values of the a vector for the gradient solutions ($N=6$)

	Case 1	Case 2	Case 3	Case 4
a_0	2.0321	1.8211	0.25009	0.20070
a_1	−1.4958	−1.0782	−0.031109	−0.072326
a_2	1.0858	0.36787	−0.87768	−0.96894
a_3	0.039546	−0.43966	0.39802	0.28644
a_4	−0.64785	−0.70931	1.2352	1.1208
a_5	12.731	11.246	187.03	204.60

Table3 Values of the a vector for the Newton solutions ($N=6$)

	Case 1	Case 2	Case 3	Case 4
a_0	1.5709	1.6880	0.25752	0.12913
a_1	−0.02688	−2.5678	−6.1155	0.54051
a_2	0.18049	13.724	44.533	−2.7785
a_3	−0.36406	−23.625	−98.798	3.7264
a_4	0.21790	10.356	66.374	−1.6260
a_5	12.726	11.118	179.61	208.74

Table 4 Values of the a vector for the Newton solutions ($N=7$)

	Case 1	Case 2	Case 3	Case 4
a_0	1.5685	1.4797	−0.19339	0.12491
a_1	0.11542	3.1722	8.6084	0.68006
a_2	−0.88232	−28.767	−69.457	−3.8163
a_3	2.5412	100.00	232.59	6.6224
a_4	−3.1033	−141.51	−332.84	−5.0088
a_5	1.3433	65.993	167.88	1.3975
a_6	12.726	11.106	186.86	208.74

The change in the performance index caused by the gradient approximate solution, however, is insignificant.

The results for case 2 are shown in Figs. 5-7 and the a vector is listed in Tables 2-4. The trajectory for case 2 looks almost like the case 1 trajectory over the first two-thirds of the time interval. The trajectory then becomes almost straight and intersects the y axis at almost 90 deg. The bank angle remains near 90 deg until late in the flight, and then the vehicle rolls up rapidly. The lift coefficient is near its maximum value early in the flight and then decreases rapidly near the end. The large values of C_L and μ initially are required to turn the vehicle quickly. After the turn is made, μ and C_L decrease. The decrease in C_L decreases the drag and hence decreases the flight time over the straight line segment.

Again the change in the performance index between the three solutions is very small. There is no consensus among the three solutions on the final value of μ or C_L. The optimal value can be determined by considering the necessary conditions for an optimal solution. If the first derivative of the Hamiltonian with respect to p is set to zero, then the necessary condition for the optimal control can be determined. This requires

$$\sin p=\frac{\lambda_\psi}{2KC_{Lh}V\lambda_V} \tag{11}$$

where λ_ψ and λ_V are the multipliers associated with the state differential equations for $\dot{\psi}$ and $\dot{V}$, respectively. For case 2, both λ_ψ and λ_V are zero at the final time. Using L'Hospital's rule, the final value of P for case 2 is

$$\sin p_f=\mathrm{ctn}\psi_f/(2KC_{Lh}) \tag{12}$$

Since ψ_f, from the Newton solutions, is approximately 270 deg, the final value of p_f should be zero. Thus the bank angle should go toward zero while C_L maintains flight in the horizontal plane or is as small as possible. This assumes, of

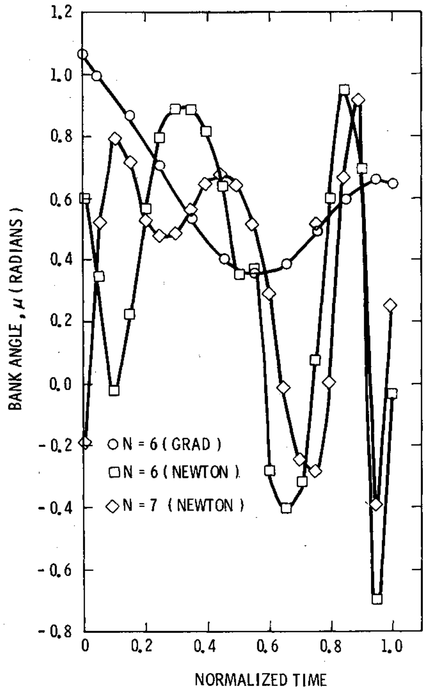

Fig. 9 Bank angle vs time for case 3.

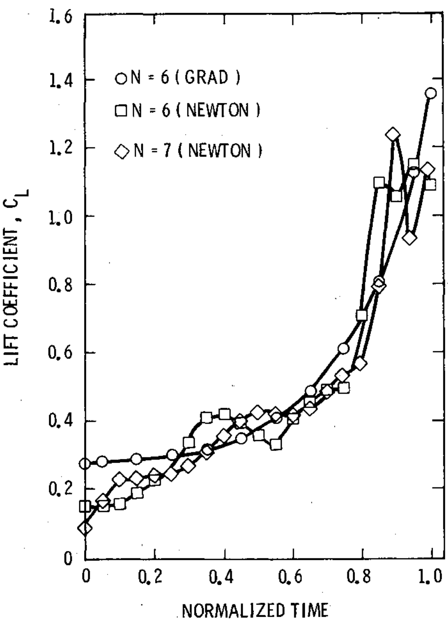

Fig. 10 Lift coefficient vs time for case 3.

course, that the final value of ψ_f from the parameterized solutions is the correct optimal value.

The change in flight time for the two minimum time trajectories is 1.5 s. The case 2 trajectory would be flown by allowing the airplane to fly approximately 1000 ft past the target before releasing the weapon. If the airplane cruises at 700 ft/s over this interval then the distance between the airplane and the target, when the vehicle reaches the target, will be about the same for the two cases. Case 2, however, does leave the weapon in the air 1.5 s less than case 1.

Case 3 is shown in Figs. 8-10. The gradient solution is a spiral return trajectory. The bank angle is initially near 65 deg. The large bank angle is required to get the vehicle to start turning back toward the origin. As the turn increases, the bank decreases. It finally increases to pull the vehicle into the origin. The lift coefficient is small initially to reduce drag.

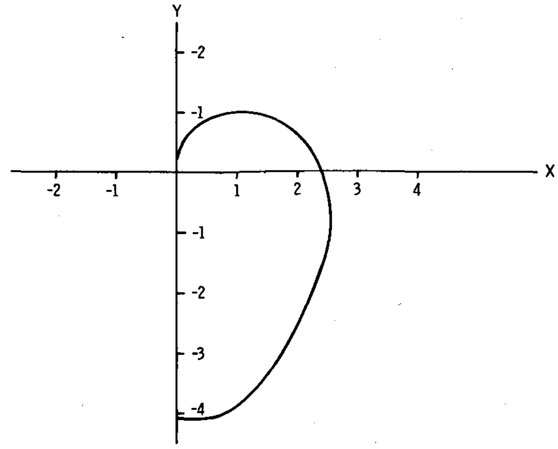

Fig. 11 Trajectory for case 4 ($N=6$, grad) (1 unit = 10,000 ft).

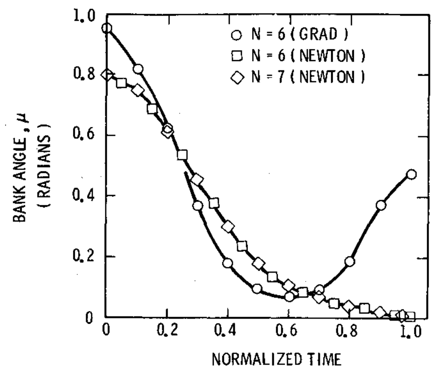

Fig. 12 Bank angle vs time for case 4.

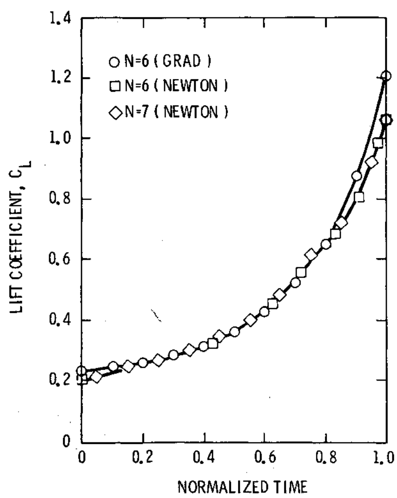

Fig. 13 Lift coefficient vs time for case 4.

Near the end of the flight it increases rapidly as the velocity decreases to maintain level flight. It is interesting to note that the end of the trajectory overlaps the start of the trajectory for this case.

The solution computed by the second-order scheme is considerably different from its starting solution, the gradient answer. This trajectory does not spiral in to the origin. It remains essentially in the first quadrant by keeping the lift coefficient and bank angle lower during the first part of the trajectory. The allows the vehicle to fly further from the origin (Fig. 8). The vehicle then turns and heads straight toward the origin. The vehicle then banks left and right during the last segment of the trajectory. The wiggles on this last segment of the trajectory are probably caused by the low-order polynomial approximation to the control. The true optimal solution probably flies a little further from the origin and then keeps the bank near zero after the turn. Over this segment it then gradually increases C_L to remain in the air as long as possible. This would be consistent with Eq. (11) which requires $\lambda_{\psi_f}=0$ for case 3. The $N=7$ solution has the same general trends as the $N=6$ solution and does show an increase in t_f.

The final value of t_f is larger for the gradient solution than it is for the $N=6$ second-order solution, which is confusing. Repeated attempts to converge a second-order solution in the vicinity of the gradient solution all failed. The gradient scheme was started near the solution of the second-order method, and it converged to a solution near the second-order one. Thus the second-order solution appears to be a minimizing solution. The reason for the failure of the second-order schemes to find a solution near the gradient solution where the value of t_f is larger than the converged second-order value is not understood.

Apparently, the low-order polynomial control approximation, in this case, is causing some reduction in the performance index since it increases by 7 s when N is increased to 7. Since t_f for case 4 is 208.74 s., however, very little increase in t_f over the $N=7$ case should be expected. Thus while there still appears to be some control history shaping problem over the last part of the trajectory, the value of the performance index generated for the $N=7$ solution is reasonable and probably is within a few percent of the optimal value.

Case 4 is shown in Figs. 11-13. The trajectory for case 4 is similar to the gradient solution for case 3 during the first half of the trajectory. During the last half, it is an almost straight line trajectory until near the end. The gradient solution then banks over sharply and intersects the y-axis at almost 90 deg. The second-order solutions keep the bank angle near zero at the end of the trajectory and, consequently, the lift coefficient remains at a lower value. Note Eq. (11) again requires $p_f=0$ for this case. The second-order solution produces a slight increase in the performance index over the gradient solution. The $N=6$ and $N=7$ second-order solutions agree very well indicating that the solution is probably a very good approximation to the true optimal trajectory.

The vehicle for case 4 would not be released over the target. It would be released approximately 40,000 ft past the target. Thus case 4 would allow the airplane to be considerably further from the target, at the time of the explosion, than case 3. When the vehicle is not required to return to the origin, the final time increases almost 20 s.

Convergence of the numerical methods is quite rapid. In general, less than 30 iterations and 20 s of CDC 6600 computer time is required per trajectory for the gradient solutions. The second-order solutions require about 10 iterations and about 15 s computer time. It is relatively easy to perform vehicle parameter studies with this approach. The results obtained seem to be reasonable and the use of polynomial approximations for the control is not expected to substantially alter the performance results obtained for the vehicle.

Acknowledgments

This work was supported by the Department of Energy.

References

[1] Hedrick, J. K. and Bryson, A. E., "Minimum Time Turns for a Supersonic Airplane at Constant Altitude," *Journal of Aircraft*, Vol. 8, March 1971, pp. 182-187.

[2] Connor, M. A., "Optimization of a Lateral Turn at Constant Height," *AIAA Journal*, Vol. 5, Feb. 1967, pp. 335-338.

[3] Bryson, A. E. and Lele, M. L., "Minimum Fuel Lateral Turns at Constant Altitude," *AIAA Journal*, Vol. 7, March 1969, pp. 559-560.

[4] Bryson, A. E., Battle, C. J., and Mikami, K., "Optimum Lateral Turns for a Reentry Glider," *Aerospace Engineering*, Vol. 21, March 1962, p. 18.

[5] Miele, A., *Flight Mechanics 1: Theory of Flight Paths*, Addison-Wesley, Reading, Mass., 1962.

[6] Williamson, W. E., "The Use of Polynomial Approximations to Calculate Sub-Optimal Controls," *AIAA Journal*, Vol. 9, May 1971, pp. 2271-2273.

[7] Fox, R. L., *Optimization Methods for Engineering Design*, Addison-Wesley, Reading, Mass. June 1973.